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GENERAL HYPERBOLIC DIFFERENCE FORMULAS FOR LINEAR AND QUASILINEAR HYPERBOLIC EQUATIONS*

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SUMMARY

The method of non-standard finite elements was used to develop multilevel difference schemes for linear and quasilinear hyperbolic equations with Dirichlet boundary conditions. A closed form equation of kth-order accuracy in space and time ($O(\Delta t^k, \Delta x^k)$) was developed for one-dimensional systems of linear hyperbolic equations with Dirichlet boundary conditions. This same equation is also applied to quasilinear systems. For the quasilinear systems a simple iteration technique was used to maintain the kth-order accuracy.

Numerical results are presented for the linear and non-linear inviscid Burger's equation and a system of shallow water equations with Dirichlet boundary conditions.

KEY WORDS Hyperbolic Non-standard Finite Elements High Order Domain of Dependence

1. INTRODUCTION

Quasilinear hyperbolic equations are of fundamental importance in fluid dynamics. Several canonical examples of such equations are (i) the inviscid Burger's equation, (ii) the shallow water (compressible gas) equations, and (iii) the Lundquist equations.

As a stepping stone to the numerical solution of the quasilinear equations with their corresponding boundary and initial data, it is reasonable to first examine linear hyperbolic equations or equivalently a linearization of the quasilinear equations. Once we have mastered the solution techniques for the linear equations we may attempt to generalize these techniques to the quasilinear setting.

There exist many high order accurate difference schemes for quasilinear hyperbolic equations, e.g. Lax and Wendroff,[1] Strang,[2–4] Rusanov,[5] Cushman and Huang,[6] Abarbanel *et al.*[7] Watanabe and Flood,[8] Fornberg,[9] Kreiss and Oliger,[10] Burnstein and Mirin,[11] and Zwas and Abarbanel.[12] In general these schemes are analysed independent of the boundary conditions, and the schemes are rarely of order higher than four. One of the major problems encountered when using high order accurate difference schemes is in the incorporation of boundary conditions with a consistent order of accuracy. For example, the high order accurate schemes of Strang[2–4] supply no insight into how one can incorporate high order accurate boundary conditions into the solution of an equation.

It is the purpose of this paper to illustrate a method which, at least in the case of Dirichlet boundary conditions for quasiliner hyperbolic equations, alleviates this problem. Before

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proceeding further we should remind the reader that it is only necessary to examine first order equations, for higher order equations can always be reduced to first order.

For quasilinear hyperbolic equations with Dirichlet boundary conditions, the technique of non-standard finite elements[6] is a viable method for developing compatible high order accurate boundary conditions. To this end we first briefly review non-standard finite elements as applied to linear hyperbolic equations.

Consider the following elementary examples:

$$u_t + cu_x = 0 \tag{1}$$

where c is a positive constant and

$$\mathbf{u}_t + \mathbf{c}\mathbf{u}_x = \mathbf{0} \tag{2}$$

where

$$\mathbf{u} = (u, v)^T \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}.$$

It is well known[13] that the characteristic data $\pm c$ determine the domain of dependence of equation (2). That is, $u(x, t)$ depends only on information contained between lines of slope $\pm 1/c$ in the (x, t)-plane below the point (x, t) (see Figure 1 (b)).

This same type of relationship holds in a curvilinear fashion for quasilinear systems (providing that the characteristics form a non-singular curvilinear co-ordinate system in a neighborhood of a point (x, t)) of hyperbolic equations (Figure 1(a)). One is thus motivated

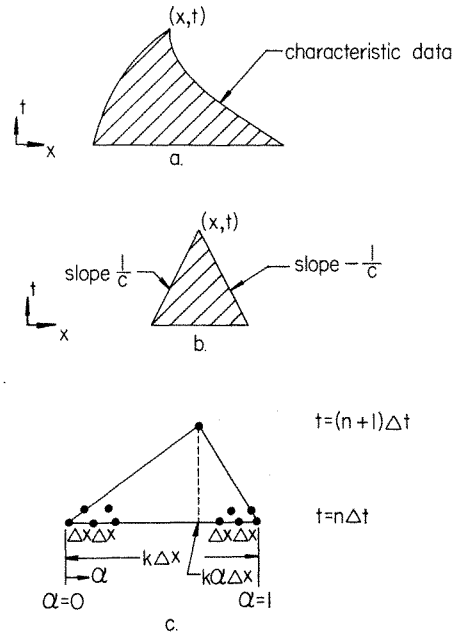

Figure 1. (a) A curvilinear data set for a quasilinear hyperbolic equation. (b) A linear characteristic data set for a linear equation. (c) The definition of the geometry factor α with kth-order polynomial interpolation

to consider developing finite element or finite difference schemes which, at any point in space-time, contain only information in and near the characteristic data sets. With this in mind it seems plausible to apply an orthogonalization method to a single element or collection of elements which contain the characteristic data. For the linear equations ((1) and (2)) it suffices to consider linear triangular elements, or more generally for the quasilinear equations one might use isoparametric triangular elements. For the sake of illustration we will use only linear triangles and consider only a single element.

We now introduce a geometry factor α defined as shown in Figure 1 (c). The parameter α controls the shape of the triangular element over which the orthogonalization method is to be applied. For each choice of α, we will define a unique finite difference scheme by applying the orthogonalization method to the triangle with geometry factor α. As we will see, a small variation in α will create (in the obvious sense) a small variation in the corresponding difference scheme, and we speculate a corresponding small change in the stability range of the schemes.

Suppose now that we apply Galerkin's method to (1) (with kth-order Lagrange interpolating polynomials as test functions) over the element in Figure 1 (c). Here of course we are assuming that the nodal spacing corresponds to that of Pascal's triangle. We thus arrive at the system of equations

$$A_{ij}u_j = 0 \tag{3}$$

where

$$A_{ij} = \int_\Omega \phi_i \left[\frac{\partial \phi_j}{\partial t} + c\frac{\partial \phi_j}{\partial x}\right] d\Omega, \tag{4}$$

u_j is the known (or unknown) value of u at node j, ϕ_i is the kth-order Lagrange interpolation polynomial, and Ω is the area of the triangular element. The reader should carefully note that we are considering only one element and hence there is no assemblage process as usually associated with finite elements.

When using kth-order interpolation over one element the Galerkin process will produce a system of $(k+1)(k+2)/2$ simultaneous equations. The linearity of the equations will depend on whether or not the underlying P.D.E. is linear. In either case we may in general eliminate from the system of equations the equations corresponding to the nodes with known nodal values of u (see Figure 2). If we assume that exactly $k+1$ nodes have known nodal values of u, then we may partition A_{ij} as follows:

$$A_{ij} = \left|\begin{array}{c:c} B_{lj} & \\ \hline C_{mn} & D_{mp} \end{array}\right| \tag{5}$$

where B_{lj} has dimension $(k+1)\times(k+1)(k+2)/2$ and corresponds to the equations with known nodal values of u, C_{mn} has dimension $(k+1)k/2\times(k+1)$ and corresponds to the remaining coefficients of A_{ij} that are multiplied by known nodal values of u, and finally D_{mp} has dimension $(k+1)k/2\times(k+1)k/2$ and corresponds to the coefficients in A_{ij} that are multiplied by the unknown nodal values of u. A simple computation will show that D_{mp} is non-singular. It is clear that the unknown nodal values of u are given by

$$u_p = -D_{mp}^{-1} C_{mn} u_n \tag{6}$$

where u_p contains the unknown nodal values of u and u_n contains the known nodal values of u. In practice we are quite often interested only in the ith value $u_p^{(i)}$ in the vector u_p. In which

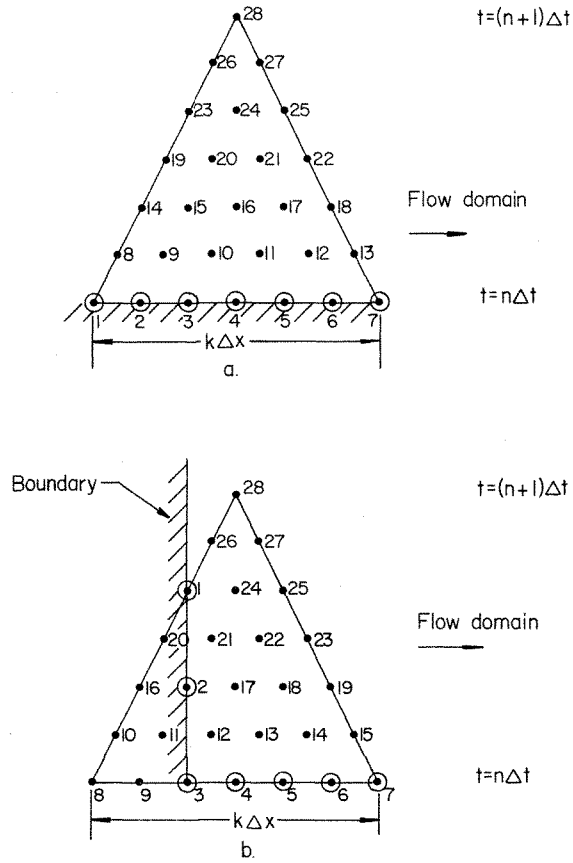

Figure 2. There are different possibilities for the location of the known nodal values depending on the initial and boundary data. Nodes 1–7 have known nodal values via boundary data, initial conditions, or previous time steps

case

$$u_p^{(i)} = -(0, \ldots, 0, 1, 0, \ldots, 0) D_{mp}^{-1} C_{mn} u_n \tag{7}$$

where the one in the p-vector of zeros is in the ith location. Equation (7) is the finite difference for the value of u at the ith node given we know the value of u at $k+1$ other nodes.

The fact that the above schemes are kth-order accurate follows from the general properties of Lagrange interpolation polynomials.[14]

2. kth-ORDER CLOSED FORM DIFFERENCE EQUATIONS FOR EQUATION (1) WITH kth-ORDER DIRICHLET BOUNDARY CONDITIONS

Using the technique discussed in Section 1 we can show in a very tedious but straightforward fashion that the value of u at node z (here z is the nodal number at which we are seeking the

value of u) determined from (6) is given by

$$u(z)=\sum_{N=1}^{k+1}\phi_N(z)u(x_N) \tag{8}$$

where

$$\phi_N(z)=\prod_{\substack{m=1\\ m\neq N}}^{k+1}\frac{F(z)-F(x_m)}{F(x_N)-F(x_m)}, \tag{9}$$

$$F(z)=(k\alpha-\nu)L_3(z)+kL_2(z) \tag{10}$$

or

$$F(z)=(k\alpha\Delta x-c\Delta t)L_3(z)+k\Delta xL_2(z),$$

α is the geometry factor, $\nu=\dfrac{c\Delta t}{\Delta x}$ is the Courant number, k is the order of polynomial interpolation, and $L_3(z)$ and $L_2(z)$ are the area co-ordinates of node z (see Figure 3(a)). The above equation holds provided $k+1$ nodes have nodal values of u corresponding to the $k+1$ numbering locations x_N, $N=1,\ldots,k+1$ (see for example Figure 3(b)).

It should be clear to the reader that the $\phi_N(z)$ are Lagrange interpolating polynomials over

$$\min_N\{F(x_N)\colon N=1,\ldots,k+1\}\leq F(z)\leq\max_N\{F(x_N)\colon N=1,\ldots,k+1\}$$

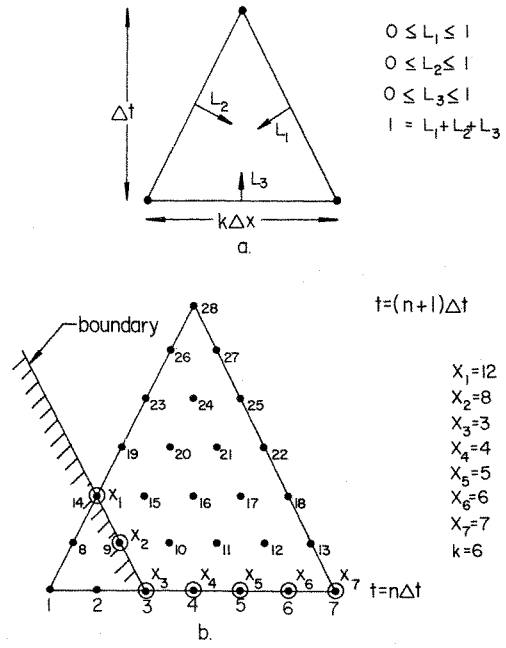

Figure 3. (a) The definitions of L_1, L_2 and L_3. (b) The $k+1$ numbering locations $\{x_N\}$ corresponding to known nodal values

where the x_N correspond to the nodes with known nodal values of u. It should also be clear that

$$\phi_N(x_m) = \delta_{Nm}.$$

We can easily show that if $F(z) = F(x_N)$ for some $N \in \{1, \ldots, k+1\}$, then $u(z)$ satisfies the shift condition, i.e. $u(z)$ is exact. Indeed,

$$u(x_N) = u^{n+L_3(x_N)}_{j+k[L_2(x_N)+\alpha L_3(x_N)]} \tag{11}$$

$$u(z) = u^{n+L_3(z)}_{j+k[L_2(z)+\alpha L_3(z)]} \tag{12}$$

and if

$$F(z) = F(x_N) \quad \left(\text{i.e.} \frac{L_2(z) - L_2(x_N)}{L_3(z) - L_3(x_N)} = \frac{\nu}{k} - \alpha\right) \tag{13}$$

then

$$\mathrm{u(z)} = u(x_N) \quad (\text{i.e. } u^n_j = u^{n+L_3(x_N)-L_3(z)}_{j+k[L_2(x_N)-L_2(z)+\alpha(L_3(x_N)-L_3(z))]}). \tag{14}$$

In the above equations j is the spatial index and n is the temporal index. The left-most node on the base of the element corresponds to u^n_j. From (13) and (14) we find

$$u^n_j = u^{n+L_3(x_N)-L_3(z)}_{j+\nu[L_3(x_N)-L_3(z)]}.$$

But $u(x,t) = f(x\text{-}ct)$ solves (1) exactly when $f(x\text{-}ct)$ is specified as an initial and/or boundary condition. Hence for our scheme to be exact we need

$$f(x_j - ct_n) = f(x_j + \Delta x\nu[L_3(x_N) - L_3(z)] - ct_n - c\Delta t[L_3(x_N) - L_3(z)])$$

but clearly

$$\Delta x\nu[L_3(x_N) - L_3(z)] = c\Delta t[L_3(x_N) - L_3(z)]$$

The proof of our claim is complete.

Let us now take a geometric look at (8)–(10). In particular let us examine (10) via Figure 4. Let L_1, L_2, and L_3 be area co-ordinates of Δafb. We construct $\overline{cz} \parallel \overline{ab}$. Since L_2 is

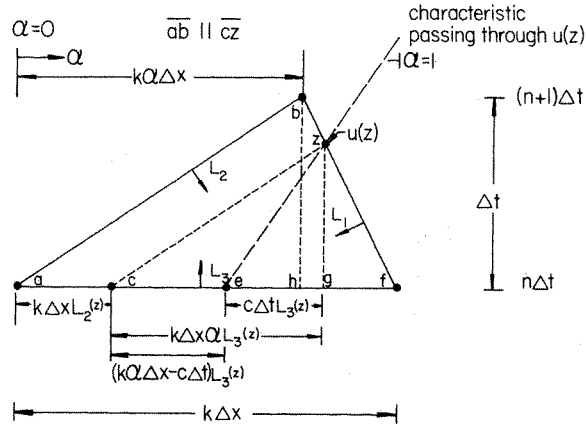

Figure 4. The geometry of the difference scheme

measured $\perp$ to $\overline{ab}$, the equation for $\overline{cz}$ is given by $L_2 = L_2(z)$ and hence $|\overline{ac}| = k\Delta x L_2(z)$ (recall that L_2 is linear and varies from zero on $\overline{ab}$ to one at f). We construct $\overline{zg}$ such that $\overline{zg} \perp \overline{af}$ which implies $|\overline{zg}| = L_3(z)\Delta t$. We construct $\overline{ez}$ by intersecting the characteristic (passing through point z) with Δafb. But the characteristic has slope $1/c$, hence $|\overline{eg}| = c\Delta t L_3(z)$. Now $\overline{bh} \perp \overline{af}$ implies $\Delta abh \sim \Delta czg$. Thus

$$\frac{|\overline{cg}|}{|\overline{gz}|} = \frac{|\overline{ah}|}{|\overline{bh}|} \Rightarrow |\overline{cg}| = k\alpha\Delta x L_3(z)$$

and hence $|\overline{ae}| = (k\alpha\Delta x - c\Delta t)L_3(z) + k\Delta x L_2(z)$. This last equation is identical to $F(z)$ of (10).

We are now in a position to see geometrically what is happening in (8). We do so by examining Figure 5. Suppose we want to use (8)–(10) to determine the value of u at node z, given that from boundary and initial data we know u at x_i, $i = 1, \ldots, 9$. Let x_i', $i = 1, \ldots, 9$ be the intercepts of the characteristic lines passing through x_i, $i = 1, \ldots, 9$, respectively, with the nth time level. In Figure 5 $x_i' = x_i$ for $i = 4, \ldots, 9$. From the previous discussion the distances from vertex a of $\Delta a x_9 z$ to the points x_i', $i = 1, \ldots, 9$ are given by $F(x_i)$, $i = 1, \ldots, 9$, respectively (since $L_3(x_i) = 0$ for $i \geq 4$ in Figure 5, $F(x_i)$ corresponds to the distance from vertex a to x_i). We thus see that the Lagrange interpolation formula (8)–(10) assigns to the points x_i', $i = 1, \ldots, 9$ the weights $u(x_i)$. The value of $u(z)$ is then determined at z' via Lagrange interpolation between the unequally spaced points x_i', $i = 1, \ldots, 9$.

Geometrically it is also clear why, when $F(z) = F(x_N)$, the scheme is exact. Indeed, if $F(z) = F(x_N)$ the characteristics passing through x_N and z are the same. Hence, since $u(x_N)$ is assumed exact, $u(z)$ must be exact.

Let us now examine (8)–(10) in more detail. In particular suppose the $k+1$ known nodal values of u are at time level n (Figure 2(a)). In this case $L_3(x_N) = 0$, $N = 1, \ldots, k+1$ where k is again the order of polynomial interpolation. Moreover if $u(x_1)$ corresponds to u_j^n, x_2 corresponds to $u_{j+1}^n, \ldots, x_{k+1}$ corresponds to u_{j+k}^n; then

$$L_2(x_N) = \frac{N-1}{k}, \qquad N = 1, \ldots, k+1.$$

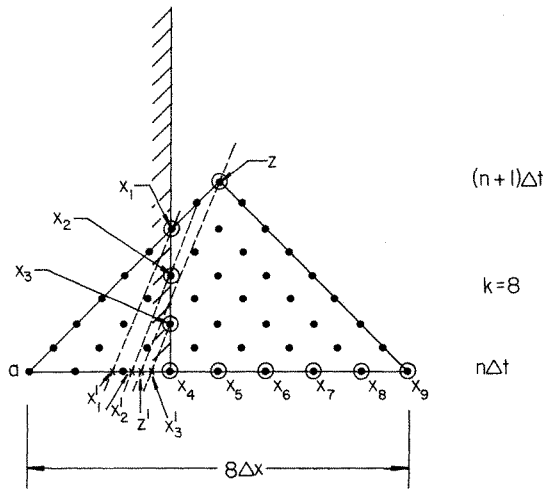

Figure 5. The characteristic explanation of the difference scheme

Thus

$$F(x_N) = N - 1.$$

Also if we let z correspond to $u_{j+\alpha k}^{n+1}$, then $L_3(z) = 1$, $L_2(z) = 0$ and hence

$$F(z) = k\alpha - \nu.$$

Equations (8)–(10) now reduce to

$$u_{j+\alpha k}^{n+1} = \sum_{N=1}^{k+1} \left[\prod_{\substack{m=1 \\ m \neq N}}^{k+1} \frac{k\alpha - \nu - 1 - m}{N - m} \right] u_{j+N-1}^{n} \tag{15}$$

which is identical to the kth-order scheme of Cushman and Huang[6]. It is also pointed out by Cushman and Huang[6] that (15) reduces to the following special cases:

(i) $k = 1$, $\alpha = 0$; unstable Euler
(ii) $k = 1$, $\alpha = \frac{1}{2}$: Lax method
(iii) $k = 1$, $\alpha = 1$: upstream differencing
(iv) $k = 2$, $\alpha = \frac{1}{2}$: Lax–Wendroff single-step

At this point it seems appropriate to comment on the stability of (8)–(10). The most obvious stability constraint is that of the C.F.L. necessity condition, i.e. the difference scheme must contain the characteristic data. One easily checks the C.F.L. criteria for a particular scheme by visual inspection.

We have not been able to establish sufficient conditions for the stability of the very general equations (8)–(10) by the von Neumann or any other method. Sufficient conditions for this general equation seem to be difficult to derive. One can of course derive sufficient conditions for specific cases of (8)–(10). For example, Cushman and Huang[6] have found that necessary and sufficient conditions for the stability of (15) are given by

$$|\nu - k\alpha + 0{\cdot}5k| \leq 1 \quad \text{if } k \text{ is even} \tag{16}$$

and

$$|\nu - k\alpha + 0{\cdot}5k| \leq 0{\cdot}5 \quad \text{if } k \text{ is odd}. \tag{17}$$

In Section 4 we will also check (at least numerically) the stability of the schemes for various boundary conditions.

3. QUASILINEAR EQUATIONS

Let us now briefly examine the more interesting system of quasilinear hyperbolic equations. Equations (8)–(10), which was developed from (1), may be extended in a very natural way to the quasilinear systems case. Consider the general quasilinear hyperbolic equation

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u}, x, t)\mathbf{u}_x = 0 \tag{18}$$

where $\mathbf{A}$ is an $l \times l$ matrix with real eigenvalues, and $\mathbf{u}$ is an $l \times 1$ column vector. To apply (8) to (18) we merely write (18) in the following discrete form:

$$\mathbf{u}_{j+k[L_3(z)+\alpha L_3(z)]}^{n+L_3(z)} = \sum_{N=1}^{k+1} \left\{ \prod_{\substack{m=1 \\ m \neq N}}^{k+1} (\mathbf{F}(z) - \mathbf{F}(x_m))(\mathbf{F}(x_N) - \mathbf{F}(x_m))^{-1} \right\} \mathbf{u}_{j+k[L_2(x_N)+\alpha L_3(x_N)]}^{n+L_3(x_N)} \tag{19}$$

where $\mathbf{F}(z)=(k\alpha\mathbf{I}-\mathbf{\Lambda}_j^n)L_3(z)+k\mathbf{I}L_2(z)$, $\mathbf{\Lambda}_j^n=(\Delta t/\Delta x)\mathbf{A}_j^n$, $\mathbf{I}$ is the 2×2 identity matrix, and $\mathbf{A}_j^n$ is an average value of $\mathbf{A}$ at the nth time level corresponding to node j.

If (18) is non-linear, then the accuracy of (19) will depend on the choice of $\mathbf{A}_j^n$. One method for evaluating $\mathbf{A}_j^n$ is to consider only the eigenvalues of the matrix corresponding to each characteristic and find the appropriate point for each eigenvalue by tracing back along the corresponding characteristic direction to find the intersection of the characteristic and the initial data line.

A viable alternative method for evaluating boundary conditions for quasilinear systems is to apply Galerkin's method to the non-linear equations themselves, rather than to a linear approximating system. This method gives rise to a system of non-linear equations of order $(k+1)k/2$ that must be solved at several nodes near the boundary. Although it is impractical to use this method everywhere in the flow domain, it is not impractical to use it near the boundaries alone. For example, one could use this method near the boundaries and use the method of Strang[4] at interior nodes and hence maintain kth-order accurate schemes.

4. NUMERICAL RESULTS

We consider first the most elementary of the linear wave equations, equation (1). If we set

$$c=0{\cdot}5,$$

$$u(x,\ 0)=\sin(x),\ x\geq 0$$

and

$$u(0,\ t)=\sin(-ct),\ t\geq 0$$

then the solution to (1) is given by $u(x,\ t)=\sin(x-ct)$.

Figure 6 represents the technique we used to evaluate the boundary condition at $x=0$. We let j be the spatial index with $j\equiv 0$ on the boundary. When $k=4$ we determine u_1^{n+1} from

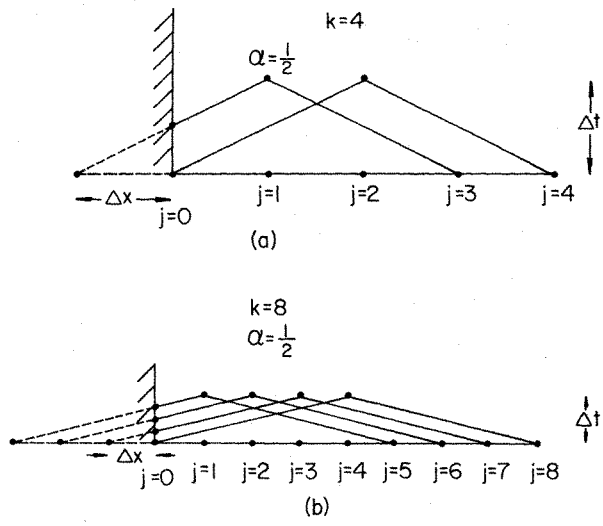

Figure 6. The orientation of the element for the vertical boundary data of the linear equation

$u_0^{n+1/2}$, u_j^n, $j=0,\ldots,3$. For $j>1$, u_j^{n+1} is determined from u_{j-i}^n and u_{j+i}^n $i=0,\ 1,\ 2$. When $k=8$ we determine u_1^{n+1} from $u_0^{n+i/4}$, $i=0,\ldots,3$, and u_j^n, $j=1,\ldots,5$. The value u_2^{n+1} is determined from $u_0^{n+i/4}$, $i=0,1,2$, and u_j^{n+1}, $j=1,\ldots,6$ while the value of u_3^{n+1} is determined from $u_0^{n+1/4}$ and u_j^n, $j=0,\ldots,7$. For $j>3$, u_j^{n+1} is determined from u_{j+i}^n and u_{j-i}^n, $i=0,\ldots,4$.

Figure 7 represents a plot of the error versus $j\Delta x$ for $k=4$ with $\Delta x=\pi/40$. In this case $\alpha=0{\cdot}5$, $\Delta t=\Delta x$, and the results are presented after 80 time steps. The error plot beyond $j\Delta x=80$ is a periodic reflection of the error between $j\Delta x=40$ and $j\Delta x=80$. Figure 8 is similar to Figure 7, however, now $k=8$. In Figure 8 the somewhat erratic nature of the error is due to the approach to machine precision (i.e. round-off error).

If we again consider (1), only we now use the data

$$c=0{\cdot}5,$$

$$u(x,\ 0)=\sin(x),\ x\geq 0$$

and

$$u(x,\ t)=\sin(x-ct)\ \text{on}\ t=-x,\ x\leq 0$$

then the solution to (1) is again $u(x,\ t)=\sin(x-ct)$. This problem presents a good illustration of the value of allowing α (which governs the geometry of the element) to vary.

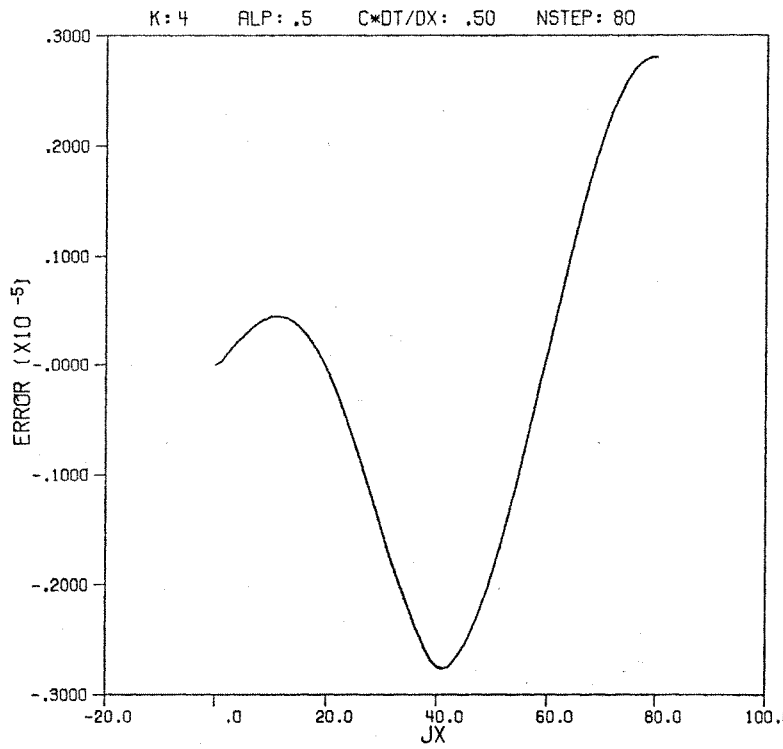

Figure 7. Numerical results for (1) with 4th-order interpolation and a vertical boundary

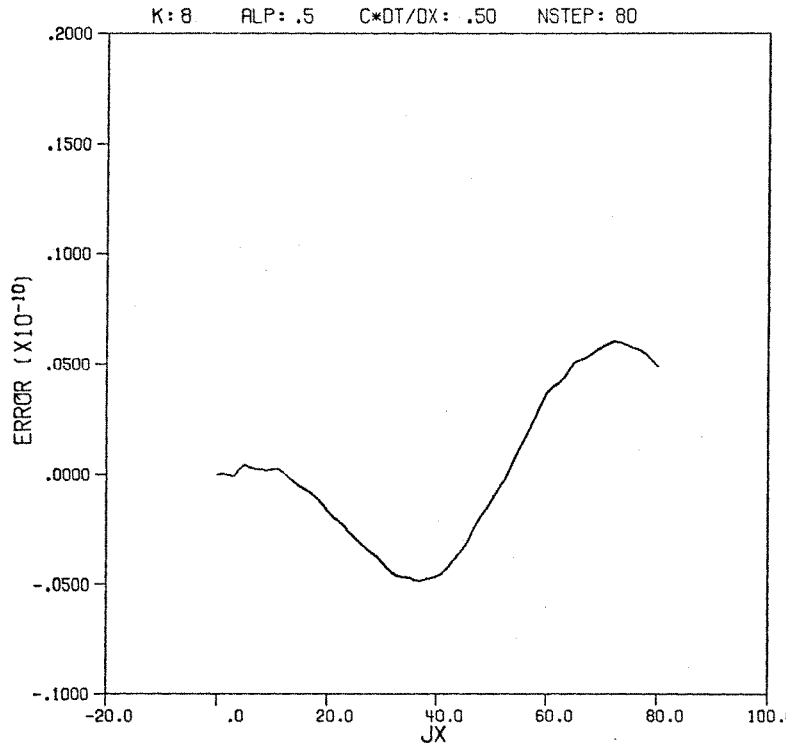

Figure 8. Numerical results for (1) with 8th-order interpolation and a vertical boundary

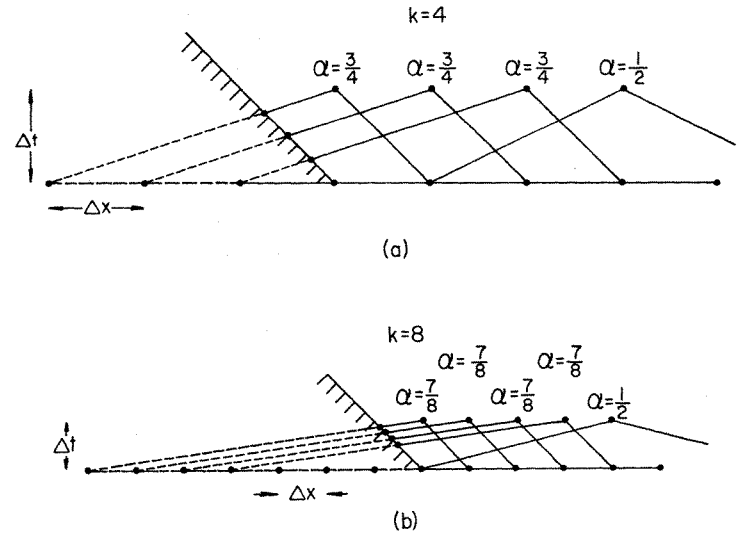

Figure 9. The orientation of the element for the slant boundary data of the linear equation

Figure 9 represents the non-standard finite element technique used to evaluate the boundary condition at $t=-x$, $x\leq 0$ on which we define the spacial index j to be identically zero. For $k=4$ and 8 we determine the value of u_1^{n+1}, u_2^{n+1}, etc. as in Figure 6, however the triangles are now oriented differently. In particular for the nodes close to the boundary we chose α such that one side of the triangle (see Figure 9) is oriented parallel to $t=-x$, $x\leq 0$. We use this orientation until we are sufficiently far from the boundary to use only values of u at the nth time level, at which time, in order to maintain stability, we shift to $\alpha=\frac{1}{2}$. Figures 10 and 11 represent plots of the error $j\Delta x$ for the above initial data, $\Delta x=\pi/40$, and $\Delta t=\Delta x$.

We now consider the more complex quasilinear inviscid Burgers' equation given by

$$u_t+uu_x=0, \qquad 1<x<2, \qquad 0<t\leq 10 \tag{20}$$

subject to

$$u(0,\ x)=(2x)^{1/2}$$

and

$$u(t,\ 1)=(t^2+2)^{1/2}-t,\ u(t,\ 2)=(t^2+4)^{1/2}-t. \tag{21}$$

The exact solution to (20) and (21) may be found in Reference 7 and is given by

$$u(t,\ x)=(t^2+2x)^{1/2}-t. \tag{22}$$

This solution is well behaved in the domain of interest so that shocks do not occur. Moreover the problem allows the use of both upstream and downstream boundary conditions. It should

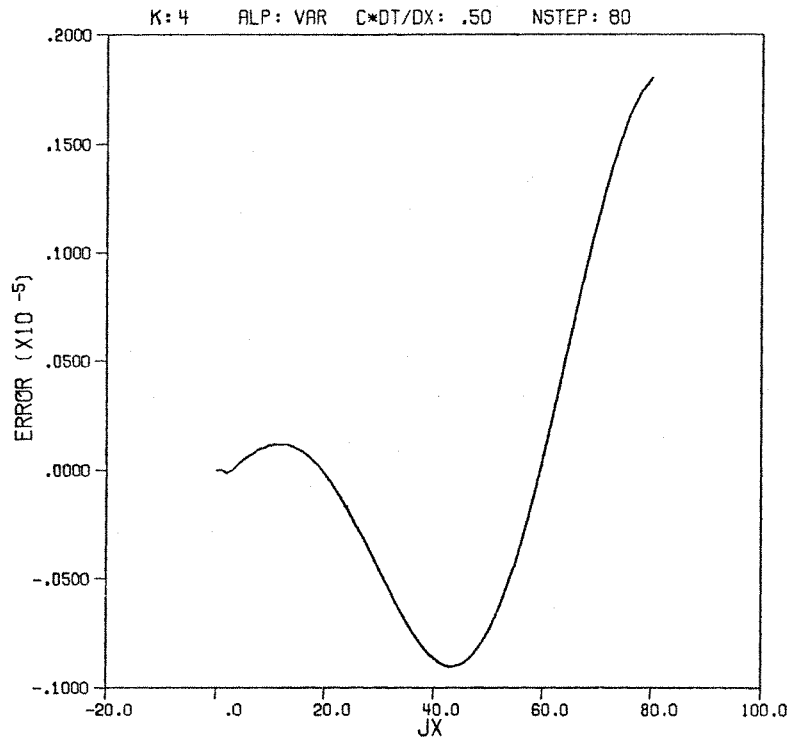

Figure 10. Numerical results for (1) with 4th-order interpolation and a slant boundary

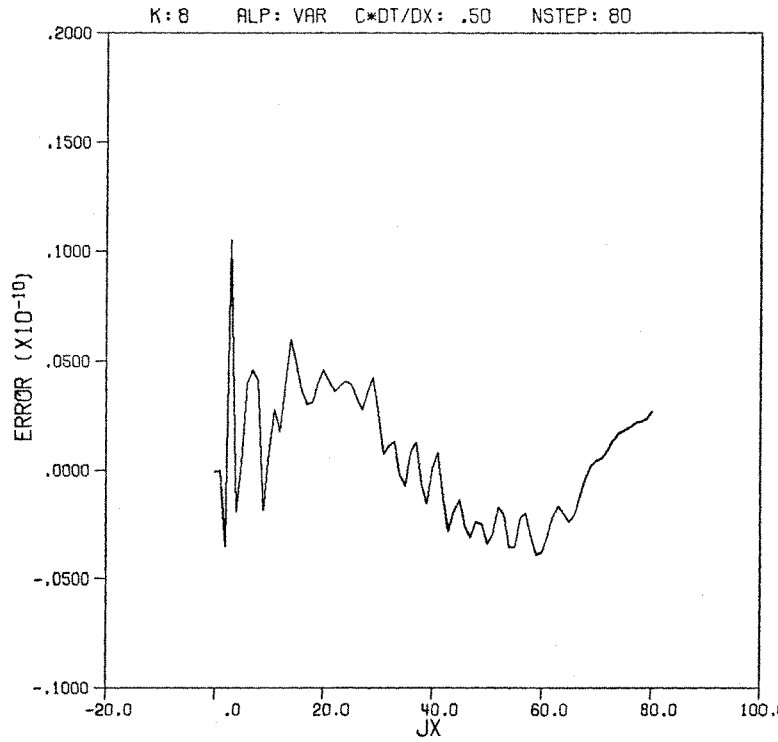

Figure 11. Numerical results for (1) with 8th-order interpolation and a slant boundary

be noted that in this case the characteristic is non-linear ($dx/dt = (t^2+2x)^{1/2} - t$). We would also like to mention that generally for physically realistic problems, one would not specify Dirichlet B.C.s at both upstream and downstream locations. However this particular problem is well posed.

As in our first problem the numerical results to follow are all with $\alpha = 1/2$ ($\alpha = 1/2$ is chosen strictly for ease in programming). Figure 12 is with quadratic interpolation, $\Delta t = 0{\cdot}025$, $\Delta x = 0{\cdot}05$, and 40 time steps. Note that in this case there is no need to introduce non-exact boundary conditions. To get the results pictured we have used the simple iteration technique of Cushman and Huang[6]. The iteration technique consists of using the predicted value of u after one step in the iteration as a guess for u at the next iteration step. To be more precise: Suppose we are trying to find u_j^{n+1}. To do so, at node j we solve

$$u_t + u_j^n u_x = 0 \tag{23}$$

(which is linear) to obtain a guess ${}^1u_j^{n+1}$ for u_j^{n+1}. We then solve

$$u_t + {}^1u_j^{n+1} u_x = 0. \tag{24}$$

In all cases, after just a few iteration steps, we have three successive guesses converging to within 10^{-11}.

Figures 13 and 14 are similar to Figure 12 but with $k = 4$ and 6, respectively. In this case the boundary conditions at $x = 1$ and $x = 2$ are handled as in the first linear problems.

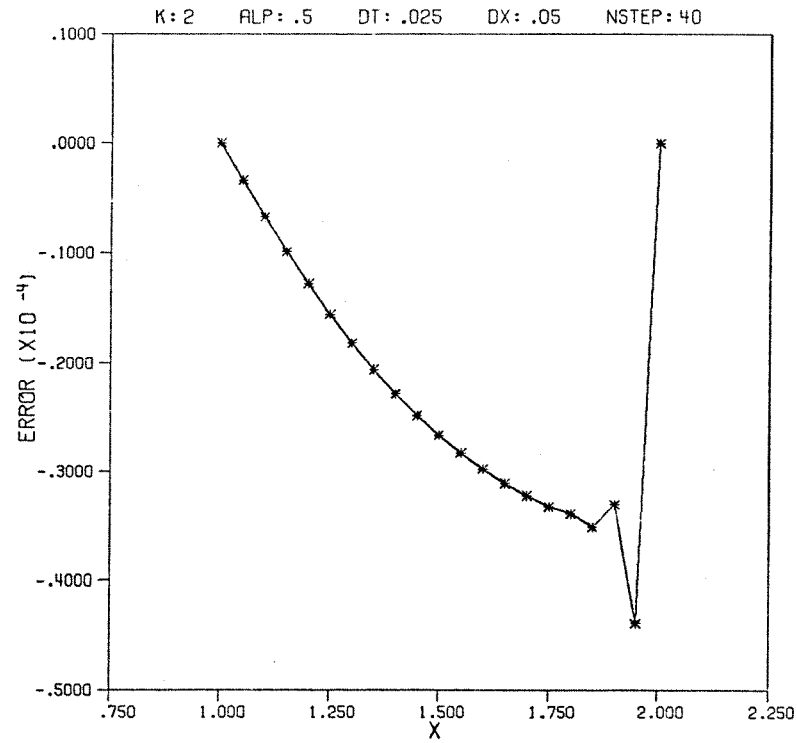

Figure 12. Quadratic interpolation of Burger's equation with iteration. Boundary conditions were evaluated exactly

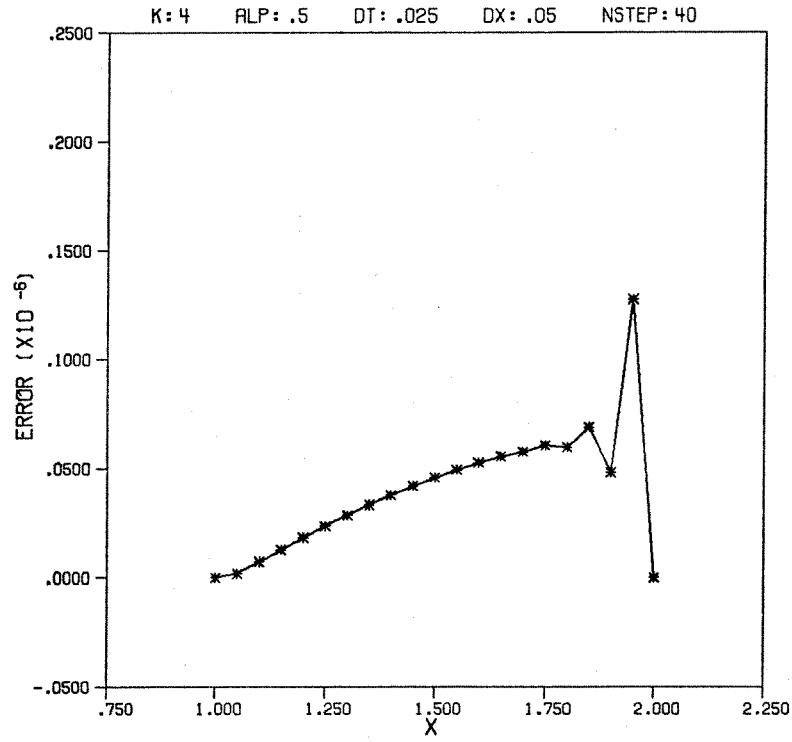

Figure 13. Quartic interpolation of Burger's equation with iteration. Boundary conditions are evaluated via (8)–(10)

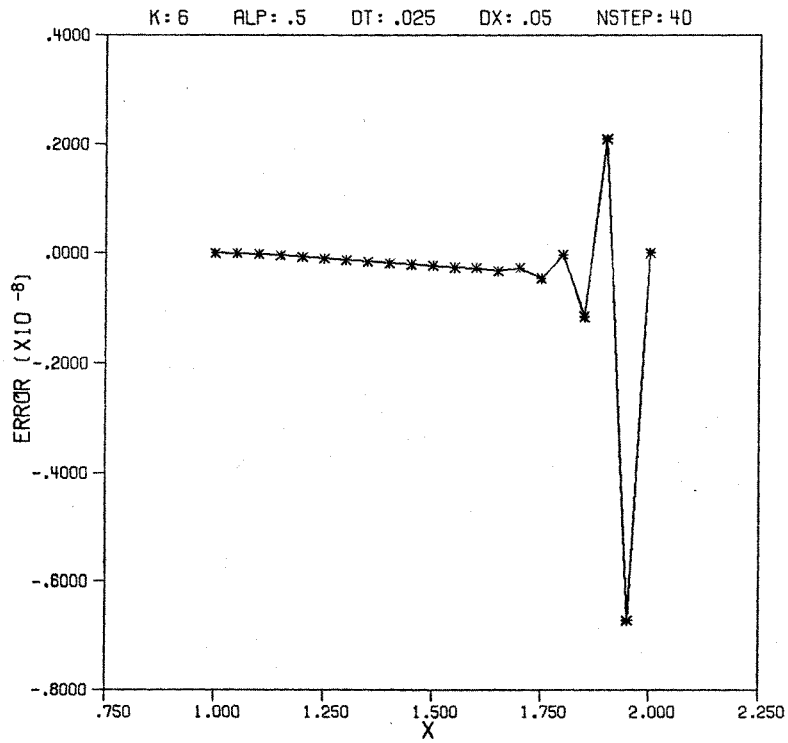

Figure 14. 6th-order interpolation of Burger's equation with iteration. Boundary conditions are evaluated via (8)–(10)

Although we have not been able to prove it analytically the numerical results for the non-linear equation suggest that the iteration produces kth-order accuracy. However, we should again point out that the numerical approximations to the linear equation can be shown analytically to be kth-order accurate (see Reference 14 or merely use the well known facts concerning Lagrange interpolation).

Let us now consider the shallow water equations. The shallow water equations with an appropriate set of boundary and initial data are given by

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \tag{25}$$

subject to

$$\mathbf{u}(x, 0) = \binom{0}{20} \text{ and } v(0, t) = 20 + t, \qquad t \geq 0, \tag{26}$$

where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ and

$$\mathbf{A} = \begin{pmatrix} u & 2v \\ \frac{1}{2}v & u \end{pmatrix}.$$

The exact solution to this problem can be found via the method of characteristics and is presented graphically in Figure 15. The boundary conditions for this problem are evaluated as in the first of the linear problems where we maintain $\alpha = \frac{1}{2}$. To use our linear difference

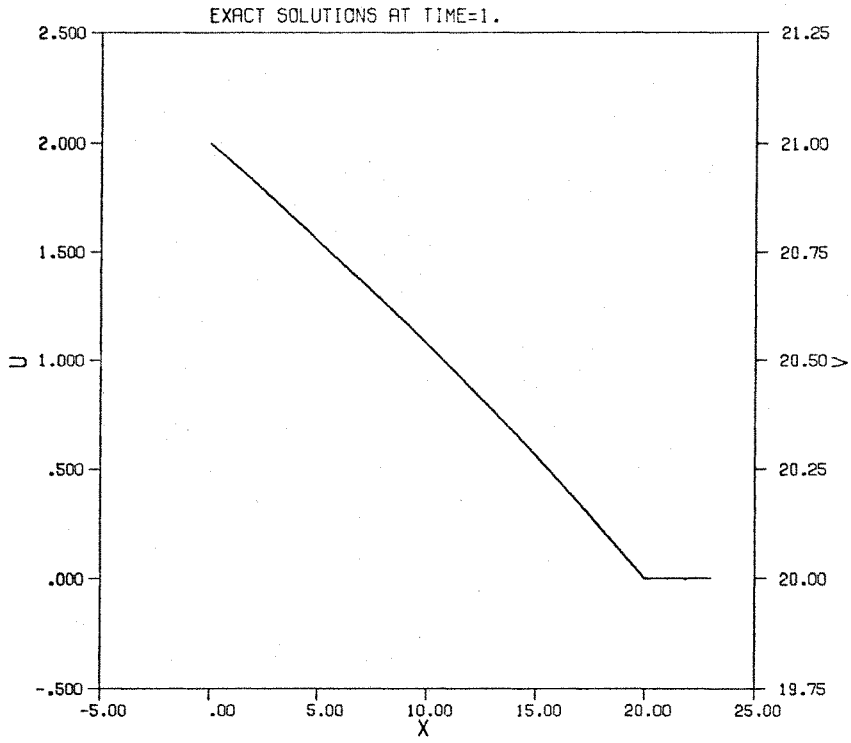

Figure 15. The exact solution of (20) for u and v at time $t = 1$

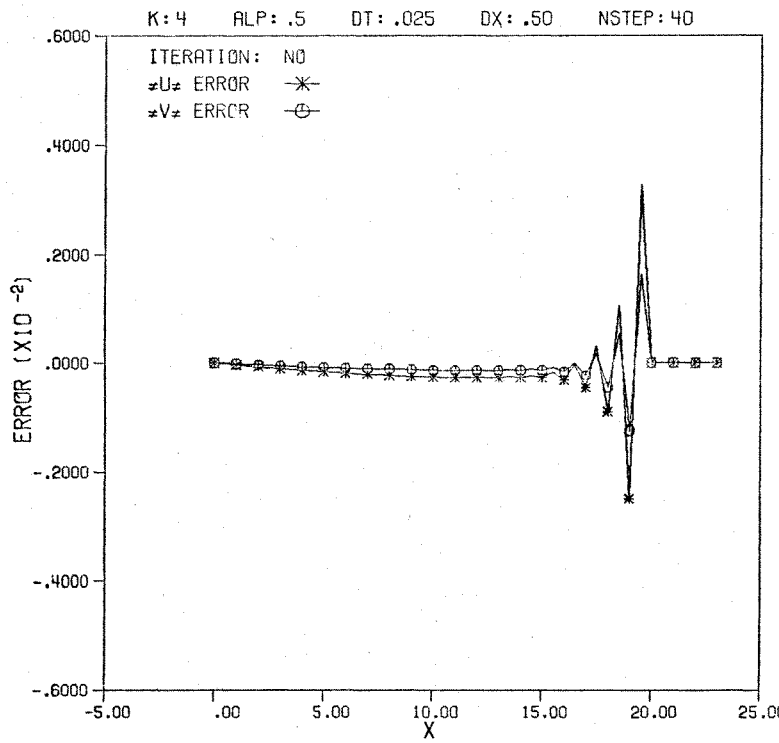

Figure 16. Numerical results for (20) with $\alpha = 1/2$ and no iteration

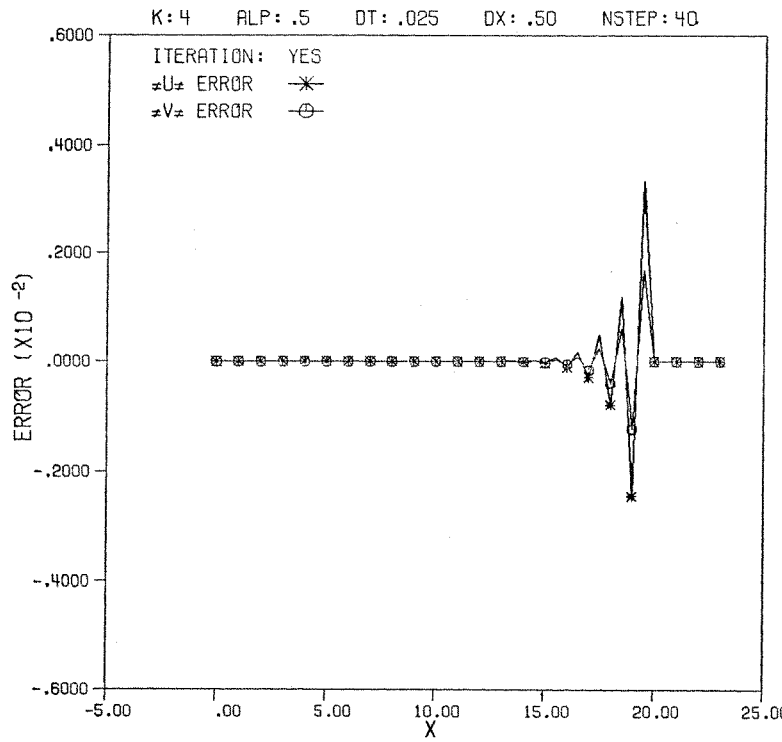

Figure 17. Numerical results for (20) with $\alpha = 1/2$ and iteration

equation on (25) we must evaluate $\mathbf{A}$ at some average $\bar{u}$ of $\mathbf{u}$. For simplicity when we were trying to find $\mathbf{u}_{j+1}^{n+1}$, we chose to evaluate $\mathbf{A}$ at $\mathbf{u} = \mathbf{u}_j^n$. With this in mind, Figure 16 is an error plot versus x for $k = 4$, $\Delta t = 0{\cdot}025$ and $\Delta x = 0{\cdot}5$ at $t = 1$. As one can see, the results are quite good with the exception of some slight dispersion at the wave front.

We were able to improve on the results presented in Figure 16 by the iteration process already mentioned. Figure 17 represents an error plot for the iterative method. Note the considerable improvement for $x \leq 15$ (the error is of the order of 10^{-10}).

To conclude this section we would like to briefly comment on stability. We already know ((16) and (17)) the necessary and sufficient conditions for the stability of the discrete version of (1) (without applying boundary conditions). We numerically tested (8)–(10) to find stability constraints when Dirichlet boundary conditions were used. We found that if (16) and (17) were satisfied in the interior, then we were able to apply the discrete boundary condition, without any stability problems. That is, the stability of (8)–(10) on the interior nodes was the major factor controlling global stability.

5. CONCLUSIONS

In this article we have presented a method for developing kth-order accurate, multilevel difference schemes for linear and quasilinear systems of hyperbolic equations with Dirichlet boundary conditions. The method is a straightforward application of the non-standard finite

element approach. This approach consists of applying Galerkin's method to a single element (with kth-order Lagrange interpolating polynomials), partitioning out the $k+1$ equations corresponding to the known nodal values (from boundary and initial data), and solving the resulting system. In a search of the literature we were unable to find any similar method for developing difference equations, although the Strang references provide a means for developing similar equations for the linear problems.

A very general difference scheme was presented for the linear inviscid Burger's equation with Dirichlet boundary conditions. Via a simple geometric argument, the scheme was shown to be similar to the method of characteristics. That is, one projects (along the characteristic) each known nodal value to a given base time, fits a kth-order (possibly unequally spaced) Lagrange polynomial through these k known values at the base time, and finally predicts the unknown we are seeking by projecting along the characteristic passing through the node (with the corresponding unknown) to its interception with the Lagrange interpolating polynomial. If

$$\frac{L_2(z)-L_2(x_N)}{L_3(z)-L_3(x_N)}=\frac{\nu}{k}-\alpha \quad \text{for some} \quad N=1,\ldots,k+1$$

the scheme was shown to satisfy the shift condition.

In a straightforward fashion we illustrated how the linear scheme could be used to solve quasilinear equations. We also presented error plots using the closed form equation for the linear and non-linear inviscid Burgers' equation and the shallow water equations.

If one wants highly accurate solutions for a specific quasilinear equation near a Dirichlet boundary, it seems reasonable to apply the non-standard finite element method in its fully non-linear form. Depending on the problem under consideration and the accuracy desired, the cost of this approach may be justified. However, this will depend very strongly on the problem under consideration. For our purposes we found it sufficient to use the linear difference schemes with iteration and thus maintain a closed form system.

One may also wish to investigate the effects of using more general curved sided isoparametric elements on curvilinear boundaries and non-linear problems.

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